Available online at www.sciencedirect.com

SCIENCE DIRECT®

European Journal of Combinatorics 27 (2006) 101–113

European Journal of Combinatorics

www.elsevier.com/locate/ejc

Poset edge-labellings and left modularity

Peter McNamara[a], Hugh Thomas[b,1]

[a] *Laboratoire de Combinatoire et d'Informatique Mathématique, Université du Québec à Montreal, Case Postale 8888, succursale Centre-ville, Montréal (Québec) H3C 3P8, Canada*
[b] *Fields Institute, 222 College Street, Toronto, ON, M5T 3J1, Canada*

Received 20 February 2004; received in revised form 14 June 2004; accepted 9 July 2004
Available online 11 September 2004

Abstract

It is known that a graded lattice of rank n is supersolvable if and only if it has an EL-labelling where the labels along any maximal chain are exactly the numbers $1, 2, \dots, n$ without repetition. These labellings are called S_n EL-labellings, and having such a labelling is also equivalent to possessing a maximal chain of left modular elements. In the case of an ungraded lattice, there is a natural extension of S_n EL-labellings, called interpolating labellings. We show that admitting an interpolating labelling is again equivalent to possessing a maximal chain of left modular elements. Furthermore, we work in the setting of an arbitrary bounded poset as all the above results generalize to this case.

MSC: 06B99; 06A11

Keywords: Supersolvable lattice; EL-labelling; Left modularity

1. Introduction

An *edge-labelling* of a poset P is a map from the edges of the Hasse diagram of P to $\mathbb{Z}$. Our primary goal is to express certain classical properties of P in terms of edge-labellings admitted by P. The idea of studying edge-labellings of posets goes back to [9].

E-mail addresses: mcnamara@lacim.uqam.ca (P. McNamara), hugh@math.unb.ca (H. Thomas).

[1] Present address: Department of Mathematics and Statistics, University of New Brunswick, Fredericton, NB, E3B 5A3, Canada.

0195-6698/$ - see front matter
doi:10.1016/j.ejc.2004.07.010

An important milestone was [2], where Björner defined EL-labellings, and showed that if a poset admits an EL-labelling, then it is shellable and hence Cohen–Macaulay. We will be interested in a subclass of EL-labellings, known as S_n EL-labellings. In [10], Stanley introduced supersolvable lattices and showed that they admit S_n EL-labellings. Examples of supersolvable lattices include distributive lattices, the lattice of partitions of $[n]$, the lattice of non-crossing partitions of $[n]$ and the lattice of subgroups of a supersolvable group (hence the terminology). It was shown in [8] that a finite graded lattice of rank n is supersolvable if and only if it admits an S_n EL-labelling. In many ways, this characterization of lattice supersolvability in terms of edge-labellings serves as the starting point for our investigations.

For basic definitions concerning partially ordered sets, see [11]. We will say that a poset P is *bounded* if it contains a unique minimal element and a unique maximal element, denoted $\hat{0}$ and $\hat{1}$ respectively. All the posets we will consider will be finite and bounded. A chain of a poset P is said to be *maximal* if it is maximal under inclusion. We say that P is *graded* if all the maximal chains of P have the same length, and we call this length the *rank* of P. We will write $x \lessdot y$ if y covers x in P and $x \,\underline{\lessdot}\, y$ if y either covers or equals x. The edge-labelling γ of P is said to be an *EL-labelling* if for any $y < z$ in P,

(i) there is a unique unrefinable chain $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$ such that $\gamma(w_0, w_1) \leq \gamma(w_1, w_2) \leq \cdots \leq \gamma(w_{r-1}, w_r)$, and
(ii) the sequence of labels of this chain (referred to as the *increasing chain* from y to z), when read from bottom to top, lexicographically precedes the labels of any other unrefinable chain from y to z.

This concept originates in [2]; for the case where P is not graded, see [3,4]. If P is graded of rank n with an EL-labelling γ, then γ is said to be an S_n *EL-labelling* if the labels along any maximal chain of P are all distinct and are elements of $[n]$. In other words, for every maximal chain $\hat{0} = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_n = \hat{1}$ of P, the map sending i to $\gamma(w_{i-1}, w_i)$ is a permutation of $[n]$. Note that the second condition in the definition of an EL-labelling is redundant in this case.

Example 1. Any finite distributive lattice has an S_n EL-labelling. Let L be a finite distributive lattice of rank n. By the fundamental theorem of finite distributive lattices [1, p. 59, Theorem 3], that is equivalent to saying that $L = J(Q)$, the lattice of order ideals of some n-element poset Q. Let $\omega : Q \to [n]$ be a linear extension of Q, i.e., any bijection labelling the vertices of Q that is order-preserving (if $a < b$ in Q then $\omega(a) < \omega(b)$). This labelling of the vertices of Q defines a labelling of the edges of $J(Q)$ as follows. If y covers x in $J(Q)$, then the order ideal corresponding to y is obtained from the order ideal corresponding to x by adding a single element, labelled by i, say. Then we set $\gamma(x, y) = i$. This gives us an S_n EL-labelling for $L = J(Q)$. Fig. 1 shows a labelled poset and its lattice of order ideals with the appropriate edge-labelling.

A finite lattice L is said to be *supersolvable* if it contains a maximal chain, called an *M-chain* of L, which together with any other chain in L generates a distributive sublattice. We can label each such distributive sublattice by the method described in Example 1 in such a way that the M-chain is the unique increasing maximal chain. As shown in [10],

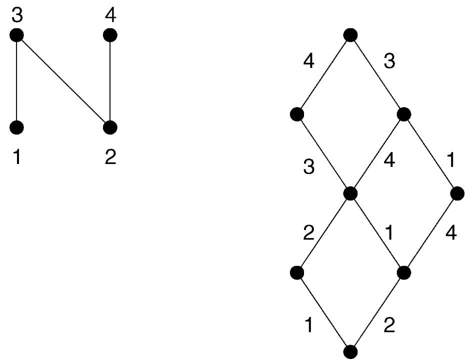

Fig. 1.

this will assign a unique label to each edge of L and the resulting global labelling of L is an S_n EL-labelling.

There is also a characterization of lattice supersolvability in terms of left modularity. Given an element x of a finite lattice L, and a pair of elements $y \leq z$, it is always true that

$$(x \vee y) \wedge z \geq (x \wedge z) \vee y. \tag{1}$$

The element x is said to be *left modular* if, for all $y \leq z$, equality holds in (1). Following Blass and Sagan [5], we will say that a lattice itself is *left modular* if it contains a left modular maximal chain, that is, a maximal chain each of whose elements is left modular. (One might guess that we should define a lattice to be left modular if all of its elements are left modular, but this is equivalent to the definition of a modular lattice.) As shown in [10], any M-chain of a supersolvable lattice is always a left modular maximal chain, and so supersolvable lattices are left modular. Furthermore, it is shown by Liu [6] that if L is a finite graded lattice with a left modular maximal chain M, then L has an S_n EL-labelling with increasing maximal chain M. In turn, as shown in [8], this implies that L is supersolvable, and so we conclude the following.

Theorem 2. *Let L be a finite graded lattice of rank n. Then the following are equivalent:*

(1) *L has an S_n EL-labelling,*
(2) *L is left modular,*
(3) *L is supersolvable.*

It is shown in [10] that if L is upper-semimodular, then L is left modular if and only if L is supersolvable. Theorem 2 is a considerable strengthening of this. Here we used S_n EL-labellings to connect left modularity and supersolvability. It is natural to ask for a more direct proof that (2) implies (3); such a proof has recently been provided by the second author in [12].

Our goal is to generalize Theorem 2 to the case when L is not graded and, moreover, to the case when L is not necessarily a lattice. We now wish to define natural generalizations of S_n EL-labellings and of left modular maxim chains.

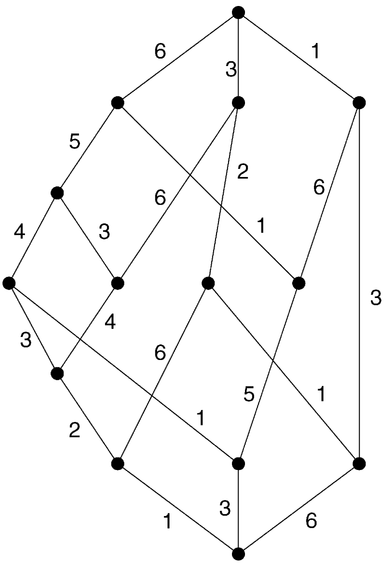

Fig. 2. The Tamari lattice T_4 and its interpolating EL-labelling.

Definition 3. An EL-labelling γ of a poset P is said to be *interpolating* if, for any $y \lessdot u \lessdot z$, either

(i) $\gamma(y, u) < \gamma(u, z)$ or
(ii) the increasing chain from y to z, say $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$, has the properties that its labels are strictly increasing and that $\gamma(w_0, w_1) = \gamma(u, z)$ and $\gamma(w_{r-1}, w_r) = \gamma(y, u)$.

Example 4. The reader is invited to check that the labelling of the non-graded poset shown in Fig. 2 is an interpolating EL-labelling. In fact, the poset shown is the so-called "Tamari lattice" T_4. For all positive integers n, there exists a Tamari lattice T_n with C_n elements, where $C_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number. More information on the Tamari lattice can be found in [4, Section 9], [5, Section 7] and the references given there, and in [6, Section 3.2], where this interpolating EL-labelling appears. The Tamari lattice is shown to have an EL-labelling in [4] and is shown to be left modular in [5].

If P is graded of rank n and has an interpolating labelling γ in which the labels on the increasing maximal chain reading from bottom to top are $1, 2, \ldots, n$, then we can check (cf. Lemma 17) that γ is an S_n EL-labelling.

Our next step is to define left modularity in the non-lattice case. Let x and y be elements of P. We know that x and y have at least one common upper bound, namely $\hat{1}$. If the set of common upper bounds of x and y has a least element, then we denote it by $x \vee y$. Similarly, if x and y have a greatest common lower bound, then we denote it by $x \wedge y$.

Now let w and z be elements of P with $w, z \geq y$. Consider the set of common lower bounds for w and z that are also greater than or equal to y. Clearly, y is in this set. If this

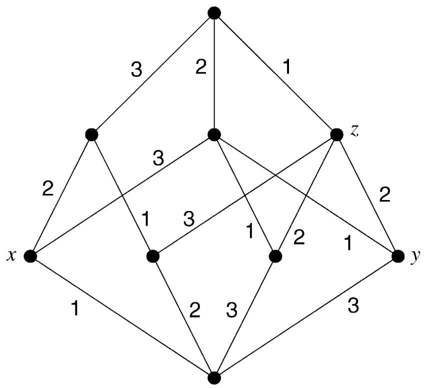

Fig. 3.

set has a greatest element, then we denote it by $w \wedge_y z$ and we say that $w \wedge_y z$ is well-defined (in $[y, \hat{1}]$). We see that $(x \vee y) \wedge_y z$ is well-defined in the poset shown in Fig. 3, even though $(x \vee y) \wedge z$ is not. Similarly, let w and y be elements of P with $w, y \leq z$. If the set $\{u \in P \mid u \geq w, y \text{ and } u \leq z\}$ has a least element, then we denote it by $w \vee^z y$ and we say that $w \vee^z y$ is well-defined in $[\hat{0}, z]$. We will usually be interested in expressions of the form $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$. The reader that is solely interested in the lattice case can choose to ignore the subscripts and superscripts on the meet and join symbols.

Definition 5. An element x of a poset P is said to be *viable* if, for all $y \leq z$ in P, $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$ are well-defined. A maximal chain of P is said to be viable if each of its elements is viable.

Example 6. The poset shown in Fig. 3 is certainly not a lattice but the reader can check that the increasing maximal chain is viable.

Definition 7. A viable element x of a poset P is said to be *left modular* if, for all $y \leq z$ in P,

$$(x \vee y) \wedge_y z = (x \wedge z) \vee^z y.$$

A maximal chain of P is said to be left modular if each of its elements is viable and left modular, and P is said to be left modular if it possesses a left modular maximal chain.

This brings us to the first of our main theorems.

Theorem 8. *Let P be a bounded poset with a left modular maximal chain M. Then P has an interpolating EL-labelling with M as its increasing maximal chain.*

The proof of this theorem will be the content of the next section. In Section 3, we will prove the following converse result.

Theorem 9. *Let P be a bounded poset with an interpolating EL-labelling. The unique increasing chain from $\hat{0}$ to $\hat{1}$ is a left modular maximal chain.*

These two theorems, when compared with Theorem 2, might lead one to ask about possible supersolvability results for bounded posets that are not graded lattices. This problem is discussed in Section 4. In the case of graded posets, we obtain a satisfactory result, namely Theorem 24. As a consequence, we have given an answer to the question of when a graded poset P has an S_n EL-labelling. This has ramifications on the existence of a "good 0-Hecke algebra action" on the maximal chains of the poset, as discussed in [8]. However, it remains an open problem to appropriately extend the definition of supersolvability to ungraded posets.

2. Proof of Theorem 8

Throughout this section, we suppose that P is a bounded poset with a left modular maximal chain $M : \hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$. We want to show that P has an interpolating EL-labelling. Our approach will be as follows. We will begin by specifying an edge-labelling γ for P such that M is an increasing chain with respect to γ. We will then prove a series of lemmas which build on the viability and left modularity properties. These culminate with Proposition 15 which, roughly speaking, gives a more local definition for γ. We will then be ready to show that γ is an EL-labelling and is, furthermore, an interpolating EL-labelling.

We choose a label set $l_1 < \cdots < l_n$ of natural numbers. (For most purposes, we can let $l_i = i$.) We define an edge-labelling γ on P by setting $\gamma(y, z) = l_i$ for $y \lessdot z$ if

$$(x_{i-1} \vee y) \wedge_y z = y \qquad \text{and} \qquad (x_i \vee y) \wedge_y z = z.$$

It is easy to see that γ is well-defined. We will refer to it as the labelling induced by M and the label set $\{l_i\}$. When P is a lattice, this labelling appears, for example, in [6,13]. As in [6], we can give an equivalent definition of γ as follows.

Lemma 10. *Suppose $y \lessdot z$ in P. Then $\gamma(y, z) = l_i$ if and only if*

$$i = \min\{j \mid x_j \vee y \geq z\} = \max\{j + 1 \mid x_j \wedge z \leq y\}.$$

Proof. That $i = \min\{j \mid x_j \vee y \geq z\}$ is immediate from the definition of γ. By left modularity, $\gamma(y, z) = l_i$ if and only if $(x_{i-1} \wedge z) \vee^z y = y$ and $(x_i \wedge z) \vee^z y = z$. In other words, $x_{i-1} \wedge z \leq y$ and $x_i \wedge z \not\leq y$. It follows that $i = \max\{j + 1 \mid x_j \wedge z \leq y\}$. $\square$

Lemma 11. *Suppose that $y \leq w \leq z$ in P and let $x \in M$. Then $((x \wedge z) \vee^z y) \vee^z w$ is well-defined and equals $(x \wedge z) \vee^z w$. Similarly, $((x \vee y) \wedge_y z) \wedge_y w$ is well-defined and equals $(x \vee y) \wedge_y w$.*

Proof. It is routine to check that, in $[\hat{0}, z]$, $(x \wedge z) \vee^z w$ is the least common upper bound for w and $(x \wedge z) \vee^z y$, and that, in $[y, \hat{1}]$, $(x \vee y) \wedge_y w$ is the greatest common lower bound for $(x \vee y) \wedge_y z$ and w. $\square$

Lemma 12. *Suppose that $t \leq u$ in $[y, z]$ and $x \in M$. Let $w = (x \vee y) \wedge_y z = (x \wedge z) \vee^z y$ in $[y, z]$. Then $(w \vee^z t) \wedge_t u$ and $(w \wedge_y u) \vee^u t$ are well-defined elements of $[t, u]$ and are equal.*

Proof. We see that, by Lemma 11,

$$\begin{aligned}(x \vee t) \wedge_t u &= ((x \vee t) \wedge_t z) \wedge_t u = ((x \wedge z) \vee^z t) \wedge_t u \\ &= (((x \wedge z) \vee^z y) \vee^z t) \wedge_t u = (w \vee^z t) \wedge_t u.\end{aligned}$$

Similarly,

$$(x \wedge u) \vee^u t = (w \wedge_y u) \vee^u t.$$

But $(x \vee t) \wedge_t u = (x \wedge u) \vee^u t$, yielding the result. □

Lemma 13. *Suppose x and w are viable and that x is left modular in P.*

(a) *If $x \lessdot w$ then for any z in P we have $x \wedge z \precapprox w \wedge z$.*
(b) *If $w \lessdot x$ then for any y in P we have $w \vee y \precapprox x \vee y$.*

Part (b) appears in the lattice case in [6, Lemma 2.5.6] and [7, Lemma 5.3].

Proof. We prove (a); (b) is similar. Assume, seeking a contradiction, that $x \wedge z < u < w \wedge z$ for some $u \in P$. Now $u \leq z$ and $u \leq w$. It follows that $u \not\leq x$.

Now $x < x \vee u \leq w$. Therefore, $w = x \vee u$. So

$$u = (x \wedge z) \vee^z u = (x \vee u) \wedge_u z = w \wedge z,$$

which is a contradiction. □

We now prove a slight extension of [6, Lemma 2.5.7] and [7, Lemma 5.4].

Lemma 14. *The elements of $[y, z]$ of the form $(x_i \vee y) \wedge_y z$ form a left modular maximal chain in $[y, z]$.*

Proof. Lemma 12 gives the viability and left modularity properties. By Lemma 13(b), $x_i \vee y \precapprox x_{i+1} \vee y$. By Lemma 12 with $z = \hat{1}$, we have that $x_i \vee y$ is left modular in $[y, \hat{1}]$. Therefore, $(x_i \vee y) \wedge_y z \precapprox (x_{i+1} \vee y) \wedge_y z$ by Lemma 13(a). □

We are now ready for the last, and most important, of our preliminary results. Let $[y, z]$ be an interval in P. We call the maximal chain of $[y, z]$ from Lemma 14 the *induced* left modular maximal chain of $[y, z]$. One way to get a second edge-labelling for $[y, z]$ would be to take the labelling induced in $[y, z]$ by this induced maximal chain. We now prove that, for a suitable choice of label set, this labelling coincides with γ.

Proposition 15. *Let P be a bounded poset, $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ a left modular maximal chain and γ the corresponding edge-labelling with label set $\{l_i\}$. Let $y < z$, and define c_i by saying*

$$\begin{aligned}y = (x_0 \vee y) \wedge_y z = \cdots &= (x_{c_1 - 1} \vee y) \wedge_y z \\ \lessdot (x_{c_1} \vee y) \wedge_y z &= \cdots = (x_{c_2 - 1} \vee y) \wedge_y z \lessdot \cdots \\ \lessdot (x_{c_r} \vee y) \wedge_y z &= \cdots = (x_n \vee y) \wedge_y z.\end{aligned}$$

Let $m_i = l_{c_i}$. Let δ be the labelling of $[y, z]$ induced by its induced left modular maximal chain and the label set $\{m_i\}$. Then δ agrees with γ restricted to the edges of $[y, z]$.

Proof. Suppose $t \lessdot u$ in $[y, z]$. Using ideas from the proof of Lemma 12,

$$\begin{aligned}\delta(t,u) = m_i \Leftrightarrow\ & (((x_{c_i-1} \vee y) \wedge_y z) \vee^z t) \wedge_t u = t \quad \text{and}\\ & (((x_{c_i} \vee y) \wedge_y z) \vee^z t) \wedge_t u = u\\ \Leftrightarrow\ & (x_{c_i-1} \vee t) \wedge_t u = t \quad \text{and} \quad (x_{c_i} \vee t) \wedge_t u = u\\ \Leftrightarrow\ & \gamma(t,u) = l_{c_i}. \quad \square\end{aligned}$$

Proof of Theorem 8. We now know that the induced left modular chain in $[y, z]$ has (strictly) increasing labels, say $m_1 < m_2 < \cdots < m_r$. Our first step is to show that it is the only maximal chain with (weakly) increasing labels. Suppose that $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$ is the induced chain and that $y = u_0 \lessdot u_1 \lessdot \cdots \lessdot u_s = z$ is another chain with increasing labels.

If $s = 1$ then $y \lessdot z$ and the result is clear. Suppose $s \geq 2$. By Proposition 15, we may assume that the labelling on $[y, z]$ is induced by the induced left modular chain $\{w_i\}$. In particular, we have that $\gamma(u_i, u_{i+1}) = m_l$ where $l = \min\{j \mid w_j \vee^z u_i \geq u_{i+1}\}$. Let k be the least number such that $u_k \geq w_1$. Then it is clear that $\gamma(u_{k-1}, u_k) = m_1$. Note that this is the smallest label that can occur on any edge in $[y, z]$. Since the labels on the chain $\{u_i\}$ are assumed to be increasing, we must have $\gamma(u_0, u_1) = m_1$. It follows that $w_1 \vee^z u_0 \geq u_1$ and since $y \lessdot w_1$, we must have $u_1 = w_1$. Thus, by induction, the two chains coincide. We conclude that the induced left modular maximal chain is the only chain in $[y, z]$ with increasing labels.

It also has the lexicographically least set of labels. To see this, suppose that $y = u_0 \lessdot u_1 \lessdot \cdots \lessdot u_s = z$ is another chain in $[y, z]$. We assume that $u_1 \neq w_1$ since, otherwise, we can just restrict our attention to $[u_1, z]$. We have $\gamma(u_0, u_1) = m_l$, where $l = \min\{j \mid w_j \geq u_1\} \geq 2$ since $w_1 \ngeq u_1$. Hence $\gamma(u_0, u_1) \geq m_2 > \gamma(w_0, w_1)$. This gives that γ is an EL-labelling. (That γ is an EL-labelling was already shown in the lattice case in [6,13].)

Finally, we show that it is an interpolating EL-labelling. If $y \lessdot u \lessdot z$ is not the induced left modular maximal chain in $[y, z]$, then let $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$ be the induced left modular maximal chain. We have that $\gamma(y, u) = m_l$ where

$$l = \min\{j \mid w_j \vee^z y \geq u\} = \min\{j \mid w_j \geq u\} = r$$

since $u \lessdot z$. Therefore, $\gamma(y, u) = m_r$. Also, $\gamma(u, z) = m_l$ where

$$l = \max\{j + 1 \mid w_j \wedge_y z \leq u\} = \max\{j + 1 \mid w_j \leq u\} = 1$$

since $y \lessdot u$. Therefore, $\gamma(y, u) = m_1$, as required. $\square$

3. Proof of Theorem 9

We suppose that P is a bounded poset with an interpolating EL-labelling γ. Let $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ be the increasing chain from $\hat{0}$ to $\hat{1}$ and let $l_i = \gamma(x_{i-1}, x_i)$. We will begin by establishing some basic facts about interpolating labellings. These results will enable us to show certain meets and joins exist by looking at the labels that appear along particular increasing chains. We will thus show that the x_i are viable. We will finish by showing that the x_i are left modular, again by looking at the labels on increasing chains.

Let $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$. Suppose that, for some i, we have $\gamma(w_{i-1}, w_i) > \gamma(w_i, w_{i+1})$. Then the "basic replacement" at i takes the given chain and replaces the subchain $w_{i-1} \lessdot w_i \lessdot w_{i+1}$ by the increasing chain from w_{i-1} to w_{i+1}. The basic tool for dealing with interpolating labellings is the following well-known fact about EL-labellings.

Lemma 16. *Let $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$. Successively perform basic replacements on this chain, and stop when no more basic replacements can be made. This algorithm terminates, and yields the increasing chain from y to z.*

Proof. At each step, the sequence of labels on the new chain lexicographically precedes the sequence on the old chain, so the process must terminate, and it is clear that it terminates in an increasing chain. □

We now prove some simple consequences of this lemma.

Lemma 17. *Let m be the chain $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$. Then the labels on m all occur on the increasing chain from y to z and are all different. Furthermore, all the labels on the increasing chain from y to z are bounded between the lowest and highest labels on m.*

Proof. That the labels on the given chain all occur on the increasing chain follows immediately from Lemma 16 and the fact that after a basic replacement, the labels on the old chain all occur on the new chain. Similar reasoning implies that the labels on the increasing chain are bounded between the lowest and highest labels on m.

That the labels are all different again follows from Lemma 16. Suppose otherwise. By repeated basic replacements, one obtains a chain which has two successive equal labels, which is not permitted by the definition of an interpolating labelling. □

Lemma 18. *Let $z \in P$ such that there is some chain from $\hat{0}$ to z all of whose labels are in $\{l_1, \ldots, l_i\}$. Then $z \leq x_i$. Conversely, if $z \leq x_i$, then all the labels on any chain from $\hat{0}$ to z are in $\{l_1, \ldots, l_i\}$.*

Proof. We begin by proving the first statement. By Lemma 17, the labels on the increasing chain from $\hat{0}$ to z are in $\{l_1, \ldots, l_i\}$. Find the increasing chain from z to $\hat{1}$. Let w be the element in that chain such that all the labels below it on the chain are in $\{l_1, \ldots, l_i\}$, and those above it are in $\{l_{i+1}, \ldots, l_n\}$. Again, by Lemma 17, the increasing chain from $\hat{0}$ to w has all its labels in $\{l_1, \ldots, l_i\}$, and the increasing chain from w to $\hat{1}$ has all its labels in $\{l_{i+1}, \ldots, l_n\}$. Thus w is on the increasing chain from $\hat{0}$ to $\hat{1}$, and so $w = x_i$. But by construction $w \geq z$. So $x_i \geq z$.

To prove the converse, observe that by Lemma 17, no label can occur more than once on any chain. But since every label in $\{l_{i+1}, \ldots, l_n\}$ occurs on the increasing chain from x_i to $\hat{1}$, no label from among that set can occur on any edge below x_i. □

The obvious dual of Lemma 18 is proved similarly:

Corollary 19. *Let $z \in P$ such that there is some chain from z to $\hat{1}$ all of whose labels are in $\{l_{i+1}, \ldots, l_n\}$. Then $z \geq x_i$. Conversely, if $z \geq x_i$, then all the labels on any chain from z to $\hat{1}$ are in $\{l_{i+1}, \ldots, l_n\}$.*

We are now ready to prove the necessary viability properties.

Lemma 20. *$x_i \vee z$ and $x_i \wedge z$ are well-defined for any $z \in P$ and for $i = 1, 2, \ldots, n$.*

Proof. We will prove that $x_i \wedge z$ is well-defined. The proof that $x_i \vee z$ is well-defined is similar. Let w be the maximum element on the increasing chain from $\hat{0}$ to z such that all labels on the increasing chain between $\hat{0}$ and w are in $\{l_1, \ldots, l_i\}$. Clearly $w \leq z$ and, by Lemma 18, $w \leq x_i$.

Suppose $y \leq z, x_i$. It follows that all labels from $\hat{0}$ to y are in $\{l_1, \ldots, l_i\}$. Consider the increasing chain from y to z. There exists an element u on this chain such that all the labels on the increasing chain from $\hat{0}$ to u are in $\{l_1, \ldots, l_i\}$ and all the labels on the increasing chain from u to z are in $\{l_{i+1}, \ldots, l_n\}$. Therefore, u is on the increasing chain from $\hat{0}$ to z and, in fact, $u = w$. Also, we have that $\hat{0} \leq y \leq u = w \leq z$. We conclude that w is the greatest common lower bound for z and x_i. □

Lemma 21. *$\hat{0} = x_0 \wedge z \leq x_1 \wedge z \leq \cdots \leq x_n \wedge z = z$, after we delete repeated elements, is the increasing chain in $[\hat{0}, z]$. Hence, $(x_i \wedge z) \vee^z y$ is well-defined for $y \leq z$. Similarly, $(x_i \vee y) \wedge_y z$ is well-defined.*

Proof. From the previous proof, we know that $x_i \wedge z$ is the maximum element on the increasing chain from $\hat{0}$ to z such that all labels on the increasing chain between $\hat{0}$ and $x_i \wedge z$ are in $\{l_1, \ldots, l_i\}$. The first assertion follows easily from this.

Now apply Lemma 20 to the bounded poset $[\hat{0}, z]$. It has an obvious interpolating labelling induced from the interpolating labelling of P. Recall that our definition of the existence of $(x_i \wedge z) \vee^z y$ only requires it to be well-defined in $[\hat{0}, z]$. The result follows. □

We conclude that the increasing maximal chain $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ of P is viable. It remains to show that it is left modular.

Proof of Theorem 9. Suppose that x_i is not left modular for some i. Then there exists some pair $y \leq z$ such that $(x_i \vee y) \wedge_y z > (x_i \wedge z) \vee^z y$. Set $x = x_i$, $b = (x_i \wedge z) \vee^z y$ and $c = (x_i \vee y) \wedge_y z$. Observe that $d := x \vee b \geq c$ while $a := x \wedge c \leq b$. So the picture is as shown in Fig. 4.

By Lemma 18, the labels on the increasing chain from $\hat{0}$ to a are less than or equal to l_i. Consider the increasing chain from a to c. Let w be the first element along the chain. If $\gamma(a, w) \leq l_i$, then by Lemma 18, $w \leq x_i$, contradicting the fact that $a = x \wedge c$. Thus the labels on the increasing chain from a to c are all greater than l_i. Dually, the labels on the increasing chain from b to d are less than or equal to l_i. But now, by Lemma 17, the labels on the increasing chain from b to c must be contained in the labels on the increasing chain from a to c, and also from b to d. But there are no such labels, implying a contradiction. We conclude that the x_i are all left modular. □

We have shown that if P is a bounded poset with an interpolating labelling γ, then the unique increasing maximal chain M is a left modular maximal chain. By Theorem 8, M then induces an interpolating EL-labelling of P. We now show that this labelling agrees with γ for a suitable choice of label set, which is a special case of the following proposition.

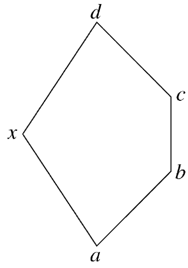

Fig. 4.

Proposition 22. *Let γ and δ be two interpolating EL-labellings of a bounded poset P. If γ and δ agree on the γ-increasing chain from $\hat{0}$ to $\hat{1}$, then γ and δ coincide.*

Proof. Let $m : \hat{0} = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = \hat{1}$ be the maximal chain with the lexicographically first γ labelling among those chains for which γ and δ disagree. Since m is not the γ-increasing chain from $\hat{0}$ to $\hat{1}$, we can find an i such that $\gamma(w_{i-1}, w_i) > \gamma(w_i, w_{i+1})$. Let m' be the result of the basic replacement at i with respect to the labelling γ. Then the γ-label sequence of m' lexicographically precedes that of m, so γ and δ agree on m'. But using the fact that γ and δ are interpolating, it follows that they also agree on m. Thus they agree everywhere. $\square$

4. Generalizing supersolvability

Suppose P is a bounded poset. For now, we consider the case of P being graded of rank n. We would like to define what it means for P to be *supersolvable*, thus generalizing Stanley's definition of lattice supersolvability. A definition of poset supersolvability with a different purpose appears in [13] but we would like a more general definition. In particular, we would like P to be supersolvable if and only if P has an S_n EL-labelling. For example, the poset shown in Fig. 3, while it does not satisfy Welker's definition, should satisfy our definition. We need to define, in the poset case, the equivalent of a sublattice generated by two chains.

Suppose P has a viable maximal chain M. Thus $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$ are well-defined for $x \in M$ and $y \leq z$ in P. Given any chain c of P, we define $R_M(c)$ to be the smallest subposet of P satisfying the following two conditions:

(i) M and c are contained in $R_M(c)$.
(ii) If $y \leq z$ in P and y and z are in $R_M(c)$, then so are $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$ for any x in M.

Definition 23. We say that a bounded poset P is *supersolvable* with M-chain M if M is a viable maximal chain and $R_M(c)$ is a distributive lattice for any chain c of P.

Since distributive lattices are graded, it is clear that a poset must be graded in order to be supersolvable. We now come to the main result of this section.

Theorem 24. *Let P be a bounded graded poset of rank n. Then the following are equivalent:*

(1) *P has an S_n EL-labelling,*
(2) *P is left modular,*
(3) *P is supersolvable.*

Proof. Observe that for a graded poset, Lemma 17 implies that an interpolating labelling is an S_n EL-labelling, and the converse is obvious. Thus, Theorems 8 and 9 restricted to the graded case give us that (1) $\Leftrightarrow$ (2).

Our next step is to show that (1) and (2) together imply (3). Suppose P is a bounded graded poset of rank n with an S_n EL-labelling. Let M denote the increasing maximal chain $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ of P. We also know that M is viable and left modular and induces the same S_n EL-labelling. Given any maximal chain m of P, we define $Q_M(m)$ to be the closure of m in P under basic replacements. In other words, $Q_M(m)$ is the smallest subposet of P which contains M and m and which has the property that, if y and z are in $Q_M(m)$ with $y \leq z$, then the increasing chain between y and z is also in $Q_M(m)$. It is shown in [8, Proof of Theorem 1] that $Q_M(m)$ is a distributive lattice. There P is a lattice but the proof of distributivity does not use this fact. Now consider $R_M(c)$. We will show that there exists a maximal chain m of P such that $R_M(c) = Q_M(m)$. Let m be the maximal chain of P which contains c and which has increasing labels between successive elements of $c \cup \{\hat{0}, \hat{1}\}$. The only idea we need is that, for $y \leq z$ in P, the increasing chain from y to z is given by $y = (x_0 \vee y) \wedge_y z \leq (x_1 \vee y) \wedge_y z \leq \cdots \leq (x_n \vee y) \wedge_y z = z$, where we delete repeated elements. This follows from Lemma 14 since the induced left modular chain in $[y, z]$ has increasing labels. It now follows that $R_M(c) = Q_M(m)$, and hence $R_M(c)$ is a distributive lattice.

Finally, we will show that (3) $\Rightarrow$ (2). We suppose that P is a bounded supersolvable poset with M-chain M. Suppose $y \leq z$ in P and let c be the chain $y \leq z$. For any x in M, $x \vee y$ is well-defined in P (because M is assumed to be viable) and equals the usual join of x and y in the lattice $R_M(c)$. The same idea applies to $x \wedge z$, $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$. Since $R_M(c)$ is distributive, we have that

$$(x \vee y) \wedge_y z = (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z) = (x \wedge z) \vee y = (x \wedge z) \vee^z y$$

in $R_M(c)$ and so M is left modular in P. $\square$

Remark 25. We know from Theorem 2 that a graded lattice of rank n is supersolvable if and only if it has an S_n EL-labelling. Therefore, it follows from Theorem 24 that the definition of a supersolvable poset when restricted to graded lattices yields the usual definition of a supersolvable lattice. (Note that this is not a priori obvious from our definition of a supersolvable poset.)

Remark 26. The argument above for the equality of $R_M(c)$ and $Q_M(m)$ holds even if P is not graded. However, in the ungraded case, it is certainly not true that $Q_M(m)$ is distributive. The search for a full generalization of Theorem 2 thus leads us to ask what can be said about $Q_M(m)$ in the ungraded case. Is it a lattice? Can we say anything even in the case that P is a lattice?

Acknowledgements

The authors would like to thank Andreas Blass, Bruce Sagan, Richard Stanley, Volkmar Welker, and the anonymous referees for helpful comments.

References

[1] G. Birkhoff, Lattice Theory, 3rd edition, American Mathematical Society, Providence, RI, USA, 1967.
[2] A. Björner, Shellable and Cohen–Macaulay partially ordered sets, Trans. Amer. Math. Soc. 260 (1980) 159–183.
[3] A. Björner, M. Wachs, Shellable non-pure complexes and posets. I, Trans. Amer. Math. Soc. 348 (1996) 1299–1327.
[4] A. Björner, M. Wachs, Shellable non-pure complexes and posets. II, Trans. Amer. Math. Soc. 349 (1997) 3945–3975.
[5] A. Blass, B. Sagan, Möbius functions of lattices, Adv. Math. 127 (1997) 94–123.
[6] L.S.-C. Liu, Left-modular elements and edge labellings, Ph.D. Thesis, Michigan State University, 1999.
[7] S.-C. Liu, B. Sagan, Left-modular elements of lattices, J. Combin. Theory Ser. A 91 (2000) 369–385. doi:10.1006/jcta.2000.3102.
[8] P. McNamara, EL-labelings, supersolvability and 0-Hecke algebra actions on posets, J. Combin. Theory Ser. A 101 (2003) 69–89. doi:10.1016/S0097-3165(02)00019-5.
[9] R. Stanley, Ordered structures and partitions, Mem. Amer. Math. Soc., vol. 119, American Mathematical Society, Providence, RI, USA, 1972.
[10] R. Stanley, Supersolvable lattices, Algebra Universalis 2 (1972) 197–217.
[11] R. Stanley, Enumerative Combinatorics, vol. 1, Wadsworth & Brooks/Cole, Monterey, CA, USA, 1986; Cambridge University Press, Cambridge, UK, 1997 (second printing).
[12] H. Thomas, Graded left modular lattices are supersolvable, 2004 (preprint), available at http://arxiv.org/math.CO/0404544.
[13] V. Welker, Shellability in the lattice of subgroups of a finite group, in: Jerusalem Combinatorics'93, Contemp. Math., vol. 178, American Mathematical Society, Providence, RI, USA, 1994, pp. 335–360.